Fourier series: An analysis of their Pointwise Convergence with an Exploration of the Gibbs Phenomenon

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1 Introduction

Have you ever wondered what music notes, electroencephalograms and the diffusion in semiconductors all have in common? Their analyses all use Fourier series!

Fourier series are a fundamental tool to study phenomena that can be described by periodic functions. In practice, they are useful for the decomposition of periodic signals such as electrical current, cerebral waves, images etc.

Let's consider a basic signal: the vibration of a tuning fork. When it vibrates, the air molecules oscillate. At a given point x at time t, the variation of the air pressure p produced by the tuning fork is a pure sinusoidal wave of pulsation $\omega = 2\pi v/\lambda$, where v is the speed of the wave and λ its wavelength. We then have $p(x,t) = A\cos(2\pi/\lambda x - \omega t)$.

If we produce simultaneously multiple sounds of different frequencies, then the resulting pressure is no longer described by a simple sinusoidal function, but by a sum of many sinusoidal functions. This is what we refer to as the superposition of waves in physics. Similarly, if we play a piano note, we do not obtain a sound wave of a unique frequency. Instead, it is the sum of a fundamental sound wave and other sound waves (called the harmonics) whose frequencies are n times the fundamental one. If $\sin(\omega t)$ and $\cos(\omega t)$ correspond to the fundamental frequency, then $\sin(n\omega t)$ and $\cos(n\omega t)$ correspond to the harmonics.

The function that describes the variation of the air pressure produced by the piano note can be potentially complicated to analyse since the wave produced is a combination of the fundamental and harmonics waves. However, this function is periodic, and its period is equal to that of the fundamental sound wave.

In fact, a periodic signal of any shape can be obtained by summing a sinusoid of a given frequency (called fundamental) with other sinusoids whose frequencies are integer multiples of the fundamental (the harmonics). To do so, it is necessary to write all the harmonics involved in the wave. This means that we need to write an *infinite sum of terms*, i.e., a series. This series is called the Fourier series.

Fourier series allow us to study a periodic function using its trigonometric decomposition, which is often easier to handle for subsequent analysis. They were introduced by Joseph Fourier in 1822. Since then, they constituted the basis of many other branches of Mathematics such as information theory and harmonic analysis.

In this paper we will study the Fourier series and how they can be used to explore the Gibbs Phenomenon. In Section 2, we start with rigorously defining the Fourier series. Section 3 then studies their convergence. Finally, in Section 4 , we explore the Gibbs Phenomenon and how Fourier series can be used to analyse it.

2 Definitions

In this section, we define the Fourier series as well as their corresponding coefficients. First, let's start with some preliminary definitions.

Definition 2.1. *Let T be a strictly positive real number. $g : \mathbb{R} \to \mathbb{C}$ is called a periodic function with period T if $g(x+T) = g(x)$ for every x in $\mathbb{R}$.*

Notice that if g is a T-periodic function, then the function $f(x) = g\left(\frac{T}{2\pi}x\right)$ is 2π-periodic. Indeed, $f(x+2\pi) = f(x)$. Consequently, one can limit themselves to the study of 2π-periodic functions.

Definition 2.2. *Let $[a,b]$ be an interval of $\mathbb{R}$. The function $f : [a,b] \to \mathbb{C}$ is piecewise continuous on $[a,b]$ if there exists a subdivision $a = a_0 < a_1 < ... < a_n = b$ and functions f_i continuous on $[a_i, a_{i+1}]$ such that f is equal to f_i on the interval (a_i, a_{i+1}).*

It is important to remark that a piecewise continuous function is not necessarily continuous on the points of subdivision. However, on these points x, it has a left limit denoted $f(x^-)$ and a right limit, $f(x^+)$. We will now define trigonometric series.

Definition 2.3. *A trigonometric series is a series of functions $\sum_{n\geq 0} f_n(x)$, where each function is of the form $f_n(x) = a_n \cos(nx) + b_n \sin(nx)$ and the a_n, b_n are real numbers for every positive integer n.*

In the below proposition, we show that a given trigonometric series can also be written with complex coefficients.

Proposition 2.4. *Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$ be a trigonometric series following definition 2.3. We can rewrite our series such that $f(x) = \sum_{n=\infty}^{\infty} c_n e^{inx}$ and each of the c_n is a complex number.*

Proof. Let $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$. We can use the identities $\cos(nx) = \frac{e^{inx}+e^{-inx}}{2}$ and $\sin(nx) = \frac{e^{inx}-e^{-inx}}{2i}$ to replace the trigonometric functions in the series with exponentials of the form e^{ikx} where k is an integer. We then get $f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ where $c_0 = \frac{a_0}{2}$, $c_n = \frac{a_n}{2} + \frac{b_n}{2i}$ and $c_{-n} = \frac{a_n}{2} - \frac{b_n}{2i}$. □

Writing a trigonometric series with complex coefficients is often times more elegant and easier to manipulate in the algebra! Let's now define our problem.

Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function. The question we are interested in answering is: under which conditions can $f(x)$ be decomposed into a series of the type

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx)), \qquad \text{where } a_n, b_n \in \mathbb{R} \text{ for all } n \tag{1}$$

or, using proposition 2.4, as a series

$$\sum_{-\infty}^{\infty} c_n e^{inx}, \qquad \text{where } c_n \in \mathbb{C} \text{ for all } n? \tag{2}$$

Such an infinite series for the periodic function f is called the *Fourier series.* [1] In order to express the coefficients a_n, b_n and c_n, which are called the *Fourier coefficients*, we need to further assume that f is bounded and piecewise continuous. The following lemma justifies why.

Lemma 2.5. *Every piecewise continuous function is Riemann-Integrable on a bounded interval.*

We will use this result without proof here. [2] In 1807, Joseph Fourier provided an analytical expression for the coefficients defined in equations (1) and (2), as will state our next definition.

Definition 2.6. *The coefficients a_n, b_n and c_n of the Fourier series are called the* Fourier coefficients*. If f is a piecewise continuous function on $\mathbb{R}$, then f is integrable on every bounded interval per lemma 2.5. We then have*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx, \quad \text{and } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

Let's prove the formula for the coefficient c_n. Note that the formulae for a_n and b_n are obtained directly from that of c_n by letting $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$.

Proof. First, we will assume that $f(x)$ indeed has a decomposition in Fourier series, and in particular that the series converges absolutely, i.e, that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ We then have

$$f(x) = \sum_{n=-\infty}^{\infty} (c_n e^{inx}).$$

We can now multiply this equality by $e^{-ikx}, k \in \mathbb{Z}$ and integrate on $[-\pi, \pi]$ piecewise. We get

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \sum_{n=\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)x}dx,$$

where we have used the fact that the Fourier series converges absolutely. [3] Furthermore,

$$\begin{cases} \int_{-\pi}^{\pi} e^{i(n-k)x}dx = \left[\frac{1}{i(n-k)}e^{i(n-k)x}\right]_{-\pi}^{\pi} = \frac{(-1)^{n-k}-(-1)^{n-k}}{i(n-k)} = 0, & \text{for every } n \neq k \\ \int_{-\pi}^{\pi} e^{i(n-k)x}dx = \int_{-\pi}^{\pi} dx = 2\pi, & \text{when } n = k. \end{cases}$$

Thus

$$\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = 2\pi c_k, \text{ for all } k.$$

□

Having an analytical expression for the Fourier coefficients allows us to define the partial sums of the Fourier series of a given function.

[1] We admit here that a given function has a unique Fourier series. The interested reader may refer to [1] for a proof.

[2] See Proposition 11.5.6 in [4] for a proof.

[3] Here we admit without proof that if the Fourier series converges absolutely, i.e. $\sum_{n=-\infty}^{\infty} |c_n| < \infty$, then $\int_a^b \lim_{N\to\infty} h_N(x)dx = \lim_{N\to\infty}\int_a^b h_N(x)dx$ where $h_N(x) = \sum_{n=-N}^{n=N} c_n e^{i(n-k)x}$. Indeed, if the Fourier series is absolutely convergent, then it converges uniformly to f. The interested reader may refer to [2] where a proof of this result is given in Corollary 2.4.

Definition 2.7. *We denote by $S_N^f(x)$ the partial sums of order N of the Fourier series for the function f,*

$$S_N^f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)) = \sum_{n=-N}^{n=N} c_n e^{inx},$$

where a_n, b_n and c_n are the Fourier coefficients of f as defined in definition 2.6.

If $\{S_N^f(x)\}_{N=1}^{\infty}$ is close enough to $f(x)$ for every x, then one could study the Fourier series instead of the function itself and hope to get accurate results. As such, it is essential to determine the conditions under which $\{S_N^f(x)\}_{N=1}^{\infty}$ converge to $f(x)$. This will be the focus of our main section.

3 Convergence of the Fourier series

In this section we probe the pointwise convergence of the Fourier series. As mentioned in Section 2, we are essentially asking the question: under which conditions does $\{S_N^f(x)\}_{N=1}^{\infty}$ converge to $f(x)$?

This is central to Fourier analysis, since it allows the analysis of an often times complicated periodic function by replacing it with its finite Fourier series. If the series converges to the original function, then the approximation is good enough, which validates asymptotically such an approach. In theory, many types of convergence could be studied (uniform convergence, convergence almost everywhere...). However, in this paper, we will only be concerned with pointwise convergence. As such, we want to define the conditions under which $S_N^f(x) - f(x) \xrightarrow[N\to\infty]{} 0$. To do so, we first need to introduce the Dirichlet Kernel whose properties are central to the convergence of the Fourier series.

3.1 The Dirichlet kernel

Definition 3.1. *We call the Dirichlet kernel, the trigonometric polynomial*

$$D_n(x) = \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} = \frac{1}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{N}\cos(nx)\right).$$

The Dirichlet kernel is essential to the study of the convergence of Fourier series. It is named after the German mathematician Peter Gustav Lejeune Dirichlet who first expressed, in 1829, the partial sums of the Fourier series using the kernel.

As we will see shortly, this Kernel has many elegant properties upon which the convergence of the Fourier series depends. The first useful property we establish relates to the area under the kernel on $[-\pi, \pi]$.

Lemma 3.2. *Let $D_N(x)$ denote the Dirichlet Kernel per definiton 3.1, then $\int_{-\pi}^{\pi} D_N(x)dx = 1$.*

Proof. First, notice that we can use the Châsles relation to get $\int_{-\pi}^{\pi} D_N(x)dx = \int_{-\pi}^{0} D_N(x)dx +$

$\int_0^\pi D_N(x)dx$. Then, by definition 3.1, we have

$$\begin{aligned}\int_0^\pi D_N(x)dx &= \int_0^\pi \frac{1}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{N}\cos(nx)\right)dx \\ &= \left[\frac{x}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{N}\frac{\sin(nx)}{n}\right]_0^\pi \\ &= \left(\frac{\pi}{2\pi}-\frac{0}{2\pi}\right)+\frac{1}{\pi}\sum_{n=1}^{N}\left(\frac{\sin(n\pi)}{n}-\frac{\sin(n0)}{n}\right)=\frac{1}{2}.\end{aligned}$$

In the above, we have used the fact that for every positive integer k, $\sin(kx)=0$ when $x=0$ or $x=\pi$. Similarly, we find that $\int_{-\pi}^{0} D_N(x)dx=\frac{1}{2}$, which then implies that $\int_{-\pi}^{\pi} D_N(x)dx=\frac{1}{2}=1$. □

The fact that the area under the Dirichlet kernel from $-\pi$ to π is equal to one will prove essential when explicitly computing the partial sums of the Fourier series. Indeed, we next show that the partial sums can be expressed as a function of the kernel.

Lemma 3.3. *Let $S_N(x)$ denote the partial sum of order N for the Fourier series as in definition 2.7. We have the following equality*

$$S_N^f(x)=\int_{-\pi}^{\pi} f(y)D_N(x-y)dy.$$

Proof. From Definition 2.7,

$$\begin{aligned}S_N^f(x) &= \sum_{n=-N}^{N} c_n e^{inx} \\ &= \sum_{n=-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy e^{inx}, && \textit{using 2.6} \\ &= \int_{-\pi}^{\pi} f(y)dy\frac{1}{2\pi}\sum_{n=-N}^{N} e^{in(x-y)} \\ &= \int_{-\pi}^{\pi} f(y)D_N(x-y)dy, && \textit{using 3.1.}\end{aligned}$$

□

Remember that the Dirichlet kernel is itself expressed as a sum per definition 3.1. As such one may wonder why we would be interested in expressing the partial sums of the Fourier series using the Dirichlet kernel, which is just another sum. In fact, while the Kernel is a somewhat awkward sum, the following result shows that we can reduce it to a single quotient of sines. Hence, this will in return allow us to compute explicitly (under certain conditions!) the partial sums.

Lemma 3.4. *Let x be a real number such that x is not an integer multiple of 2π. The Dirichlet kernel at x is given by*

$$D_N(x)=\frac{1}{2\pi}\left(\frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}\right).$$

Proof. For a real number x which is not an integer multiple of 2π, by definition 3.1, we have

$$\begin{aligned}
D_N(x) &= \frac{1}{2\pi}\sum_{n=-N}^{N} e^{inx} \\
&= \frac{1}{2\pi} e^{-iNx}(1 + e^{ix} + ... + e^{i2Nx}) \\
&= \frac{1}{2\pi} e^{-iNx}\frac{e^{i(2N+1)x} - 1}{e^{ix} - 1} \\
&= \frac{1}{2\pi}\frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1} \\
&= \frac{1}{2\pi}\frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} \\
&= \frac{1}{2\pi}\frac{\sin((N+\frac{1}{2})x}{\sin(\frac{x}{2})}.
\end{aligned}$$

□

Notice that if x is an integer multiple of 2π, we can then use 3.1 to compute $D_N(x) = \frac{1}{\pi}\left(\frac{1}{2} + N\right)$. Lemmas 3.2-3.4 constitute the main properties of the Dirichlet Kernel that we will use later on. They are powerful in the sense that they relate the convergence of an infinite series to the limit of a certain sequence of integrals. It substantially simplifies the study of convergence since working with and estimating integrals is usually much easier than the corresponding problem with infinite series.

Before moving on to study the pointwise convergence of the Fourier series we need to state a final powerful result relating the properties of the function f to the convergence of its Fourier coefficients.

3.2 Riemann-Lebesgue Lemma

Lemma 3.5. *Let $f : [a, b] \to \mathbb{C}$ be a Riemann-integrable function on $[a, b]$. Then,*

$$\begin{aligned}
\lim_{n\to\infty}\int_a^b f(x)\cos(nx)dx = 0 \\
\lim_{n\to\infty}\int_a^b f(x)\sin(nx)dx = 0 \\
\lim_{n\to\pm\infty}\int_a^b f(x)e^{-inx}dx = 0.
\end{aligned}$$

Riemann first presented this theorem as part of his thesis on trigonometric series in 1854. The result can be shown, though not easily, by proving that every integrable function is a kind of limit of differentiable functions and then taking limits. An important consequence of the Riemann-Lebesgue Lemma is given by the following corollary.

Corollary 3.6. *If f is a 2π-periodic function continuous on $[-\pi, \pi]$, then,* $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}|b_n| = \lim_{n\to\pm\infty}|c_n| = 0$.

We will use this result without proof. [4] It is powerful since it tells us that the Fourier coefficients (as defined in 2.6) of a 2π-periodic and piecewise continuous function go to zero as n goes to infinity. Thus, one can foresee that it will be essential when deriving the conditions of pointwise convergence of the Fourier series to its function.

The properties of the Dirichlet Kernel derived in this section and the Riemann-Lebesgue lemma will be our main tools to characterise the conditions for pointwise convergence. We do so next.

3.3 Pointwise convergence of the Fourier series

Remember that we are interested in determining when $\{S_N(x)\}_{N=1}^{\infty}$ converge to $f(x)$ pointwise on $[-\pi, \pi]$. An answer to this question is found in the below theorem which was originally stated by Dirichlet.

Theorem 3.1. *(Dirichlet, 1824). Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function that is piecewise smooth. Then, for every x,*

$$\lim_{N\to\infty} S_N(x) = \frac{f(x-) + f(x+)}{2}, \textit{ where } f(x\pm) = \lim_{h\to 0, h>0} f(x \pm h).$$

In particular $\lim_{N\to\infty} S_N(x) = f(x)$ on every point x where f is continuous.

The conditions of Dirichlet require f to be piecewise smooth. This means that on any real closed interval $[a, b]$, both f and f' are piecewise continuous. In particular, $f'(a+)$ and $f'(b-)$ exist. We will not prove Dirichlet original theorem. [5] Instead we will prove the below theorem for continuously differentiable functions.

Theorem 3.2. *Let f be a 2π-periodic function. If f is continuously differentiable on $[-\pi, \pi]$, then $\{S_N(x)\}_{N=1}^{\infty}$ converges pointwise to $f(x)$ on $[-\pi, \pi]$.*

Proof. From Lemma 3.3 we can write

$$\begin{aligned} S_N(x) &= \int_{-\pi}^{\pi} f(y) D_N(x-y)dy && (3)\\ &= \int_{-\pi}^{\pi} f(x-u) D_n(u)du && (4)\\ &= \int_{-\pi}^{\pi} f(x+v) D_n(v)dv. && (5) \end{aligned}$$

In the above, the second equality is obtained by the change of variables $u = x-y$ using the fact that the integrated function is 2π-periodic. Similarly, the third equality is obtained by the change of variables $v = -u$ and using the fact that $\sum_{k=-n}^{n} e^{ikv} = \sum_{k=-n}^{n} e^{ikv}$.

Let $x \in [-\pi, \pi]$. Since f is continuously differentiable at x we can write

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} f(x+y) D_n(y)dy - f(x). \tag{6}$$

[4] The interested reader may refer to [8] for a proof.
[5] The interested reader may refer to Chapter 2 of [3].

Recall from Lemma 3.2 that $\int_{-\pi}^{\pi} D_N(y)dy = 1$. Thus,

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} f(x+y)D_n(y)dy - \int_{-\pi}^{\pi} D_N(y)dyf(x) \tag{7}$$
$$= \int_{-\pi}^{\pi} \left[(f(x+y) - f(x))\, D_N(y)dy\right]. \tag{8}$$

Now recall from Lemma 3.4 that $D_N(y) = \frac{1}{2\pi}\left(\frac{\sin((N+\frac{1}{2})y)}{\sin(\frac{y}{2})}\right)$. Notice that this expression is not defined when $\sin(\frac{y}{2}) = 0$, which happens when y is an integer multiple of 2π. Thus, in the interval $[-\pi,\pi]$, when $y = 0$, we define $D_N(0) = \frac{1}{\pi}\left(\frac{1}{2} + N\right)$. For every non-zero y, we thus get

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} (f(x+y) - f(x))\, \frac{1}{2\pi}\left(\frac{\sin((N+\frac{1}{2})y)}{\sin(\frac{y}{2})}\right) dy. \tag{9}$$

Let $g_x(y) = \frac{f(x+y)-f(x)}{2\sin(\frac{y}{2})}$. g is piecewise continuous on $[-\pi, 0) \cup (0, \pi]$. Since f is assumed to be continuously differentiable on $[-\pi,\pi]$, we have $\lim_{y\to 0} \frac{f(x+y)-f(x)}{y} = f'(x)$. Consequently we get that

$$g_x(y) = \frac{f(x+y) - f(x)}{y} \frac{\frac{y}{2}}{\sin\left(\frac{y}{2}\right)} \xrightarrow[y\to 0]{} f'(x),$$

which is well defined since f is continuously differentiable on $[-\pi,\pi]$. Thus, g is continuous on $[-\pi,\pi]$ where we define $g_x(0) = f'(x)$. We can rewrite (9) such that

$$S_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} g_x(y) \sin\left[\left(N + \frac{1}{2}\right) y\right] dy. \tag{10}$$

Using our trigonometric identities we can expand $\sin\left(\left(n+\frac{1}{2}\right)y\right) = \sin(Ny)\cos(1/2) + \sin(1/2)\cos(Ny)$. Plugging this back in (10), we get

$$S_N(x) - f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[g_x(y)\sin(Ny)\cos(y/2) + g_x(y) cos(Ny) sin(y/2\right] dy. \tag{11}$$

We let $c_x(y) = g_x(y)\cos(y/2)$ and $d_x(y) = g_x(y)\sin(y/2)$. It follows by definition of $g_x(0)$ that $c_x(0) = 0$ and $d_x(0) = 0$. We then have

$$S_N(x) - f(x) = \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} c_x(y)\sin(Ny)dy}_{\text{Fourier coefficient } b_N \text{ for } c_x} + \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} d_x(y)\cos(Ny)dy}_{\text{Fourier coefficient } a_N \text{ for } d_x} \quad \textit{by } 2.6. \tag{12}$$

We now prove that c_x is continuous and 2π-periodic. We have shown that g is continuous on $[-\pi,\pi]$. Furthermore $h : y \to \cos(y/2)$ is continuous on $[-\pi,\pi]$. Thus c_x is continuous on $[-\pi,\pi]$.

For every $y \neq 0$ in $[-\pi,\pi]$,

$$\begin{aligned} c_x(y+2\pi) &= \frac{f(x+y+2\pi) - f(x)}{2\sin(y/2+\pi)}\cos(y/2+\pi) \\ &= \frac{f(x+y) - f(x)}{2\sin(y/2)}\cos(y/2), \qquad \textit{since } f, \cos \textit{ and } \sin \textit{ are } 2\pi \textit{ periodic} \\ &= c_x(y). \end{aligned}$$

By definition of g, we also have 2π-periodicity when $y = 0$. Thus, $c_x(y)$ is 2π-periodic. Similarly, we have that $d_x(y)$ is continuous on $[-\pi, \pi]$ and 2π-periodic.

We can now apply the Riemann-Lebesgue Lemma to conclude. Let a_N and b_N be defined as in (12), Then by 3.6, $\lim_{N\to\infty} |a_N| = \lim_{N\to\infty} |b_N| = 0$. Thus $\lim_{N\to\infty} S_N(x) - f(x) = 0$. $\square$

We now have an answer to our question! We know that the pointwise convergence on a bounded interval of a Fourier series to its original function happens when the function is continuously differentiable on that interval and 2π-periodic! More generally, referring to the conditions of Dirichlet stated in Theorem 3.1, pointwise convergence obtains when the function is piecewise smooth and 2π-periodic. This is an extremely important result as it will allow us, in practice, to replace the analysis of a complicated periodic function by its finite Fourier series (which is often much easier to handle!). Given the initial function is piecewise smooth, we know that our approximation will be asymptotically close to the original function.

4 Understanding the Gibbs Phenomenon

Under Theorem 3.1, one would expect that the graph of $S_N^f(.)$ converges to the graph of f as $N \to \infty$ since the series of f converges pointwise to $\frac{1}{2}(f(x-) + f(x+))$ for every point x on the domain of f. However this is not always true! In a 1899 Nature paper, J. Willard Gibbs showed that when the function f has discontinuity points, the graph of S_N^f displays oscilliations around the points of discontinuity: this is the *Gibbs Phenomenon*. In fact, for the graph of S_N^f to converge towards the graph of $f(.)$ as $N \to \infty$, we need a stronger form of convergence : *Uniform Convergence*. This type of convergence is not satisfied on the whole domain of defintion for Fourier series whose functions have discontinuities. Let's now explain the Gibbs Phenomenon. To do so we will take the example of the 'square wave' of height $\pi/4$.

4.1 Example: The square wave

In the below figure, we present a graph of the square wave.

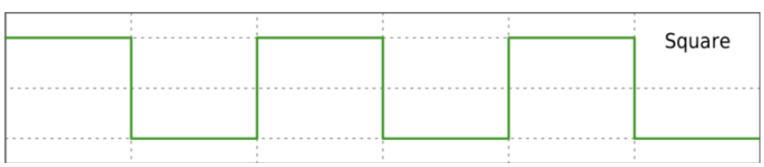

Figure 1: Graph of a square wave. *Source-Wikipedia*

For every positive integer n we define f to be the function

$$f(x) = \begin{cases} \frac{\pi}{4} & \text{if } x \in [2n\pi, (2n+1)\pi] \\ -\frac{\pi}{4} & \text{if } x \in [(2n+1)\pi, (2n+2)\pi]. \end{cases}$$

Notice that f has a jump discontinuity of height $\pi/4 - (-\pi/4) = \pi/2$ at every multiple of π. Without loss of generality, let N be even (the case where N is odd is similar). Then, the corresponding Fourier series for the square wave is

$$S_N^f(x) = \sin(x) + \frac{1}{3}\sin(3x) + ... + \frac{1}{N-1}\sin((N-1)x).$$

Below, we plot the graph of S_N^f as we increase N.

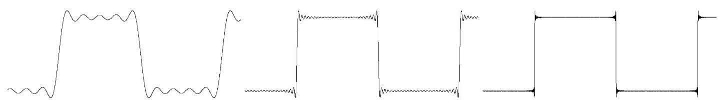

Figure 2: The graph of S_N^f as we increase N to the right. *Source: Wikipedia*

Notice that despite N increasing, the amplitude of overshoots and undershoots at the discontinuity points does not decrease. Let's analyse the discontinuity point $x = 0$ for a fixed N. On the one hand we have that

$$\begin{aligned} S_N^f(0) &= \sin(0) + \frac{1}{3}\sin(0x) + ... + \frac{1}{N-1}\sin((N-1)) \\ &= 0 = \frac{\frac{\pi}{4} - \frac{\pi}{4}}{2} = \frac{f(0-) + f(0+)}{2}. \end{aligned}$$

On the other hand, we can calculate the value of the finite sum S_N^f at a point close but greater than 0. We then get

$$\begin{aligned} S_n^f(0 + \frac{2\pi}{N}) &= \sin(\pi/N) + \frac{1}{3}\sin(3\pi/N) + ... + \frac{1}{N-1}\sin((N-1)\pi) \\ &= \frac{\pi}{2}\left[\frac{2}{N}sinc\left(\frac{1}{N}\right) + \frac{2}{N}sinc\left(\frac{3}{N}\right) + ... + \frac{2}{N}sinc\left(\frac{N-1}{N}\right)\right], \end{aligned}$$

where $sinc(x) = \frac{\sin(\pi x)}{\pi x}$. This then implies that $\lim_{N\to\infty} S_n^f\left(0 + \frac{2\pi}{N}\right) = \frac{\pi}{2}\int_0^1 sinc(x)dx$. Thus,

$$\begin{aligned} \lim_{N\to\infty} S_n^f\left(0 + \frac{2\pi}{N}\right) &= \frac{\pi}{2}\int_0^1 \frac{\sin(\pi x)}{\pi x}dx \\ &= \frac{1}{2}\int_0^1 \frac{\sin(\pi x)}{x}dx \\ &= \frac{\pi}{4} + \frac{\pi}{2}\cdot(0.0899490). \end{aligned}$$

Similarly, we can use the fact the the sinus function is odd to get

$$\lim_{N\to\infty} S_n^f\left(0 - \frac{2\pi}{N}\right) = \lim_{N\to\infty} -S_n^f\left(\frac{2\pi}{N}\right) = -\frac{\pi}{4} - \frac{\pi}{2}\cdot(0.0899490).$$

Here, the important fact to notice is that, not only do the partial sums overshoot (resp. undershoot) the point of discontinuity on the right (resp. left), but these overshoot and undershoot do not die out as N gets larger! In fact, for any value of N, the graph of S_N^f will have a total overshoot 9% of of the magnitude of the jump! Let's now characterise the Gibbs Phenomenon more generally.

4.2 The Gibbs Phenomenon in general

If f is a continuous function with period T with a discontinuity point at x_0, we then have

$$\lim_{x\to x_0+} f(x) - \lim_{x\to x_0-} f(x) = f(x_0+) - f(x_0-) = a \neq 0,$$

where a is the difference between the two limits of the function on the right and on the left of the discontinuity point.

Remember that the Finite sum for the Fourier series is

$$S_N^f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(\frac{2\pi nx}{T}) + b_n \sin(\frac{2\pi nx}{T})).$$

Then,

$$\lim_{N\to\infty} S_n^f(x_0 + \frac{T}{2N}) = f(x_0+) + a(0.089)$$

and

$$\lim_{N\to\infty} S_n^f(x_0 + \frac{T}{2N}) = f(x_0-) - a(0.089),$$

even though,

$$\lim_{N\to\infty} S_n^f(x_0) = \frac{f(x_0-) + f(x_0+)}{2}.$$

Hence, while the Fourier partial sums S_N^f converge pointwise to the function f, which means the approximations gets better as N gets larger, the overshoot and undershoot persist at the points of discontinuity.

References

[1] On The Uniqueness Problem for Fourier Series, A, Vagharshakyan, Institute of Mathematics Academy Sciences of Armenia, Bagramian, 2003.

[2] Cnvergence of Fourier series, S. Xue.
`http://math.uchicago.edu/~may/REU2017/REUPapers/Xue.pdf`

[3] Fourier Analysis and Related Topics, J. Korevaar.
`https://staff.science.uva.nl/j.korevaar/Foubook.pdf`

[4] Analysis I, T. Tao, Chapter 11, 2006.
`https://math.unm.edu/~crisp/courses/math402/spring19/TaoChapter11AnalysisI-1.pdf`

[5] Foundations of Mathematical Analysis, R. Johnsonbaugh & W.E. Pfaffenberger, 1981.

[6] A Study of The Gibbs Phenomenon in Fourier series and Wavelets, K. Raeen, 2008.

[7] Wikipedia - Gibbs Phenomenon.
`https://en.wikipedia.org/wiki/Gibbs_phenomenon`

[8] https://www.math.cuhk.edu.hk/course_builder/1415/math3060/Chapter\%201.\%20Fourier\%20series\%20.pdf

[9] Theorie Analytique de la Chaleur, J. Fourier, 1822.

[10] J. W. Gibbs, Fourier's series, letter in Nature 59 (1899), 606.

[11] A. Zygmund, Trigonometric series, 2nd ed., Cambridge Univ. Press, Cambridge, 1959.